Protecting Target Zone Currency Markets from Speculative Investors

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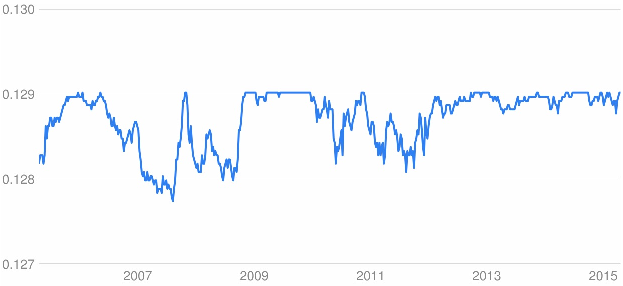
Joint work with Alexander Schied

Currency Target Zone Markets

1. An exchange rate allowed to move inside a specific regime with one or more barriers that are enforced by monetary authorities.

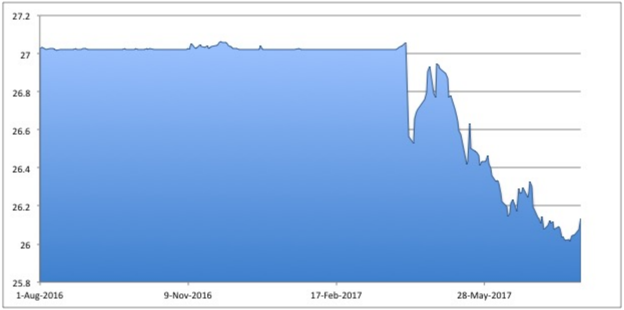
2. Examples:
   - European Exchange Rate Mechanism (before the Euro).
   - HKD vs. USD
   - CHF vs. EUR
   - CZK vs. EUR.

Currency target zone: USD/HKD



USD/HKD exchange rate from 2007 until 2015.

Currency target zone: EUR/CZK



EUR/CZK exchange rate August 2016 to August 2017.

Currency target zones–types of players:

1. The central bank aims to enforce the target zone.
2. Investors trading against the central bank.
3. Investors trading in the same direction as the central bank.

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In this talk will consider all types of players.

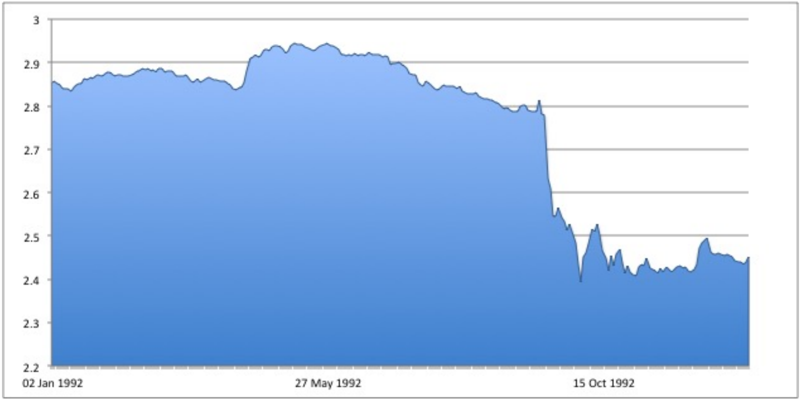
– We first describe the central bank vs. investors trading against it as a Stackelberg game, for which we find equilibrium.

Currency target zones–types of players:

1. The central bank aims to enforce the target zone.
2. Investors trading against the central bank.
3. investors trading in the same direction as the central bank.

Trading in the same direction was studied in N. and Schied (2016).

Existing literature: Krugman (1991), Svensson (1991), Bertolla and Caballero (1992), Jong (1994) ...



GBP/DEM exchange rate from January 1, 1992 to December 31, 1992.

Why target zones may fail ?

1. GBP/DEM: In September 16 of 1992, George Soros shorted 10 billion pounds within a single day. The British treasury could not keep up with the purchases and announced an exit from the ERM.
2. CHF/EURO: By 2014 the Swiss national bank spent about $\$480$ billion-worth of foreign currency on this policy, which is 70% of Swiss GDP (the Economist, Jan. 2018).
3. The reason is the inventory in currency accumulated by the central bank strategy.

The central bank

1. The goal of the central bank is to keep the exchange rate above a certain level c.
2. This is achieved by buying the domestic currency (e.g GBP) in exchange to foreign currency (e.g DEM).
3. Purchases create market impact which derives the exchange rate upwards.

Central bank vs. trader game

1. Assume that the unaffected currency exchange rate $S = (S_t)_{t\geq 0}$ is a Brownian motion.

2. Let X_t be the amount of currency that was liquidated by the speculative trader at time t.

3. The actual currency exchange rate, affected by the permanent market impact of the trader,

$$S_t^X := S_t + \gamma X_t,$$

where $\gamma > 0$ is a constant.

The central bank's reaction

1. Let R_t^X be the amount of currency which was purchased by the central bank up to time t, in reaction to strategy X of the trader.

2. Then we expect R_t^X to satisfy

$$S_t^X := S_t + \gamma X_t + \gamma R_t^X \geq c, \quad \text{for all } t \geq 0.$$

3. Here S^X incorporates the market impact of the trader and the central bank, hence this is the visible price process in the market.

The admissible strategies of both players

1. Trader: Let $\mathcal{V}$ denote the class of all continuous functions $v(t,x)$ which are locally Lipschitz and have at most linear growth in the x-variable.

2. Central bank: Let $\mathscr{R}^v$ be the class of all continuous non-decreasing functions $\{R_t^v\}_{t\geq 0}$ with $R_0^v = 0$, which satisfy for all $t \geq 0$,

$$S_t^v := S_t + \gamma X_t^v + \gamma R_t^v \geq c.$$

The admissible strategies of both players

1. Trader: Let $\mathcal{V}$ denote the class of all continuous functions $v : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ which are globally Lipschitz continuous in the spatial variable.

2. Bank: Let $\mathscr{R}^v$ be the class of all continuous non-decreasing functions $\{R_t^v\}_{t\geq 0}$ with $R_0^v = R_0$, which satisfy for all $t \geq 0$,

$$S_t^v := S_t + \gamma X_t^v + \gamma R_t^v \geq c.$$

3. Here v is interpreted as the trading speed of the trader, i.e.

$$X_t^v = \int_0^t v(r, S_r^v)\, dr.$$

The central bank's minimal inventory strategy

Theorem (N. and Schied, 2018)

For any trader admissible strategy $v \in \mathcal{V}$ there exists a unique minimal element in $\bar{R}^v \in \mathscr{R}^v$, which is given by

$$\bar{R}^v_t = \frac{1}{\gamma} L^c_t(S^v), \quad t \geq 0,$$

where $L^c(S^v) = \{L^c_t(S^v)\}_{t \geq 0}$ is the local time of S^v at c.

Idea of the proof:

- By an extension of the Skorokhod lemma to reflected SDEs we show that $L_t^c(S^v)$ is the minimal element which keeps the equation

$$S_t^v := S_t + \gamma \int_0^t v(r, S_r^v)\, dr + L_t^c(S^v).$$

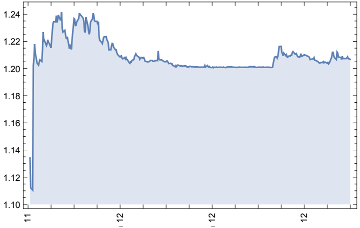
  above c for all $t \geq 0$.

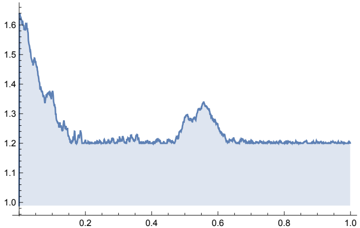
- For any other admissible strategy R^v we have $\frac{1}{\gamma} L_t^c(S^v) \leq R_t^v$, for all $t \geq 0$.

Consequences of the preceding theorem

1. Establishes optimal response for the central bank.
2. Characterizes the Skorokhod reflection term as a pathwise minimizer.
3. Provides an argument why it is natural to model exchange rates in target zones by reflected diffusions (this was assumed in earlier papers).

Modelling target zones with reflected diffusions





EUR/CHF exchange (left), reflected geometric Brownian motion (right)

The trader's control problem

1. The trader tries to maximize the central bank optimal purchase policy $\bar{R}^v$.

2. Since $\bar{R}^v$ depends linearly on $L^c(S^v)$, which is an increasing process, an equivalent goal for the trader is to maximaize $E[L^c(S^v)]$.

3. The trader also creates a temporary market impact which creates additional costs
$$\kappa \int_0^T v(t, S_t^v)^2 \, dt,$$
for $\kappa > 0$.

The trader's control problem

Therefore, the goal of the trader is to maximize (over $v \in \mathcal{V}$),

$$E_z\Big[L^c_T(S^v) - \kappa \int_0^T v(t, S^v_t)^2 \, dt\Big].$$

where

$$S^v_t := S_t + \gamma \int_0^t v(r, S^v_r) \, dr + L^c_t(S^v).$$

The log-Laplace transform for the local time

Let $\beta = \gamma^2/(2\kappa)$. For any $z \geq c$ define

$$U(t, z) = \frac{1}{\beta} \log \Big(E_z \Big[\exp \Big(\beta L_t^c \Big) \Big] \Big),$$

where L_t^c is the local time of the Brownian motion S at level c.

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We have closed-form expression for U,

$$U(t,z) = \frac{1}{\beta}\log\left(\mathrm{erf}\Big(\frac{z-c}{\sigma\sqrt{2t}}\Big) + e^{-\beta(z-c)+\beta^2\sigma^2 t/2}\Big[1 - \mathrm{erf}\Big(\frac{z-c}{\sigma\sqrt{2t}} - \frac{\beta\sigma\sqrt{t}}{\sqrt{2}}\Big)\Big]\right)$$

where erf is the Gaussian error function.

Solution to the trader's control problem

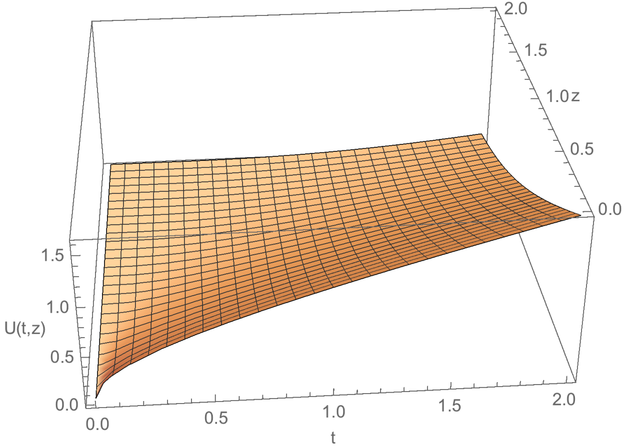
Theorem (N. and Schied, 2018)

Let $z \geq c$. Then we have

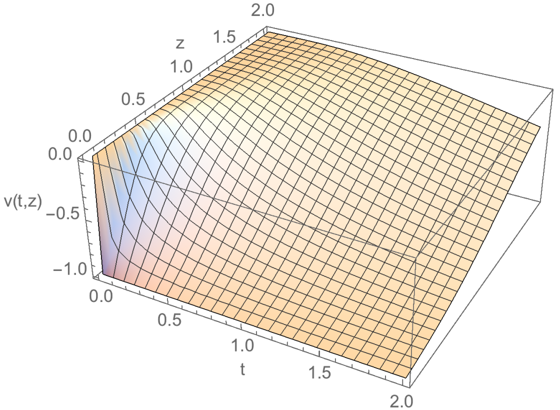
$$U(t,z) = \sup_{v \in \mathcal{V}} E_z\Big[L_t^c(S^v) - \kappa \int_0^t v(r, S_r^v)^2 \, dt \Big].$$

Moreover, there exists a unique strategy $v^* \in \mathcal{V}$ for which the supremum is attained. It is given by

$$v^*(t,x) = \frac{\gamma}{2\kappa} \partial_z U(T-t, x).$$

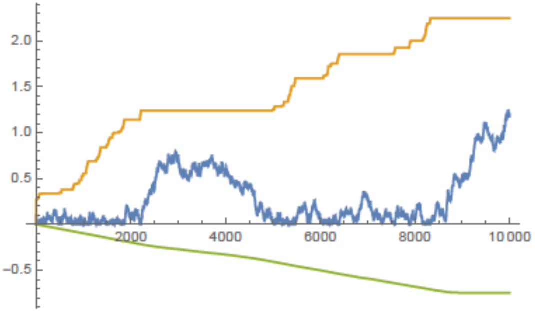


The value function $U(t, z)$ for $\sigma = \gamma = \kappa = 1$ and $c = 0$



The optimal strategy $v^*(t, z)$ for $\sigma = \gamma = \kappa = 1$ and $c = 0$

A simulation of the Stackelberg equilibrium



exchange rate S (blue)
central bank inventory (orange)
inventory of the strategic investor (green)

Value function of the trader:

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where erf is the Gaussian error function.

How to guess the value function for the trader's problem

1. We study a regularized version of the trader's problem.

2. We define
$$G_\varepsilon(x) := \frac{1}{\sqrt{2\pi\varepsilon}} e^{-(x-c)^2/(2\varepsilon)},$$
and
$$L_t^{c,\varepsilon}(S^v) = \int_0^t G_\varepsilon(S_r^v)\, dr.$$

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3. Then we let

$$U^\varepsilon(t,z) := \sup_{v \in \mathcal{V}} E_z\Big[L_t^{c,\varepsilon}(S^v) - \kappa \int_0^t v(t, S_t^v)^2\, dt\Big].$$

How to guess the value function for the trader's problem

Standard heuristic arguments suggest that the function U^ε should solve the HJB equation,

$$\partial_t U^\varepsilon = \frac{1}{2}\partial_{zz} U^\varepsilon + G_\varepsilon + \sup_{v\in\mathbb{R}}(\gamma v \partial_z U^\varepsilon - \kappa v^2),$$

with the initial condition,

$$U^\varepsilon(0,z) = 0 \quad \text{for all } z \in \mathbb{R},$$

and boundary condition

$$\partial_z U^\varepsilon(t, c+) = 0 \quad \text{for all } 0 \le t \le T. \tag{0.1}$$

How to guess the value function for the trader's problem

1. Let $\beta = \gamma^2/2\kappa$. Then U^ε is given by

$$U^\varepsilon(t,z) = \frac{1}{\beta}\log E_z\Big[\, e^{\beta\int_0^t G_\varepsilon(S_r)\,dr}\,\Big].$$

How to guess the value function for the trader's problem

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$$U^\varepsilon(t,z) = \frac{1}{\beta} \log E_z\Big[e^{\beta \int_0^t G_\varepsilon(S_r)\, dr} \Big].$$

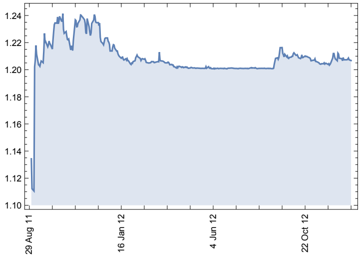
2. Then we show that for any $t \geq 0$

$$U^\varepsilon(t,z) \to U(t,z), \quad \text{as } \varepsilon \to 0, \quad \text{uniformly in } z,$$

3. where U is the value function for the trader's problem,

$$U(t,z) = \frac{1}{\beta} \log \Big(E_z \Big[\exp\Big(\beta L_t^c \Big) \Big] \Big).$$

Trading in the same direction as the central bank



An investor wishing to sell Swiss Francs will do it during the period of a lower bound of the EUR/CHF exchange rate.

The price process (reminder)

1. $S = \{S(t)\}_{t \geq 0}$ is a diffusion process with $S(0) = z$, reflected at some barrier $c \in \mathbb{R}$.
2. $L = \{L_t\}_{t \geq 0}$ is the local time of S at the barrier c.

The class of controls

1. **Trading speed:** let $\mathscr{X}$ denote the class of all progressively measurable control processes ξ for which $\int_0^T |\xi_t| \, dL_t < \infty$ P_z-a.s.

2. **Inventory:** for $\xi \in \mathscr{X}$ and $x_0 \in \mathbb{R}$ we define

$$X_t^{\xi} := x_0 + \int_0^t \xi_s \, dL_s, \; t \geq 0.$$

Quantifying the Trading Costs

We consider the minimization of the cost functional for some $p \geq 2$,

$$E_z\Big[\int_0^T |\xi_t|^p \, L(dt) + \int_0^T \phi(S_t)|X_t^\xi|^p \, dt + \varrho(S_T)|X_T^\xi|^p\Big]$$

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The Control Problem of the Trader

We consider the minimization of the cost functional for some $p \geq 2$,

$$E_z\left[\int_0^T |\xi_t|^p \, L(dt) + \int_0^T \phi(S_t) |X_t^\xi|^p \, dt + \varrho(S_T) |X_T^\xi|^p \right]$$

1. ϕ is a bounded measurable function.
2. $\varrho \geq 0$ is a bounded continuous penalty function.

What is a superprocesses ?

1. Define $\{S_t^i\}_{i=1}^{N(t)}$ - a collection of critical branching diffusion particles that live in $\mathbb{R}$.

2. $N(t)$ is the number of the particles in the system at time t.

3. We assume that between branching events the particles follow independent diffusion paths which are independent.

4. Critical branching means that each particle splits into two or dies with equal probability (independently of other particles).

What is a superprocesses ?

5. We assume that the times between branching are independently distributed exponential random variables with mean $\mathbf{1/m}$.

6. In what follows $\boldsymbol{m}$ is "large" (fast branching), $\boldsymbol{N(0) \sim m}$.

What is a superprocesses ?

1. We define the following measure valued process

$$Y_t^{(m)}(A) = \frac{1}{m} \sum_{i=1}^{N(t)} \delta_{S_t^{(i)}}(A), \quad A \subset \mathbb{R}.$$

   Here δ_x is the delta measure centred at x.

2. Suppose that $\{Y_0^m\}_{m \geq 1}$ converges weakly to μ, as $m \to \infty$.

3. In the appropriate topology, $\{Y_t^m\}_{t \geq 0}$ converges weakly to a limiting process $\{Y_t\}_{t \geq 0}$, which is called superporcess.

What is a Catalytic-Superprocesses ?

(with a single point catalyst at c)

We assume that the probability that a particle survives between $[r, t]$ and dies between $[t, t + dt]$ is given by

$$e^{-L(r,t)} dL(t),$$

where $\{L(t)\}_{t \geq 0}$ is the local time that the particle spends at the point c between $[0, t]$.

The Control Problem for a Small Trader

We consider the minimization of the cost functional for some $p \geq 2$,

$$E_z\Big[\int_0^T |\xi_t|^p \, L(dt) + \int_0^T \phi(S_t)|X_t^\xi|^p \, dt + \varrho(S_T)|X_T^\xi|^p\Big]$$

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Unique Solution to the Control Problem

1. Consider the catalytic superprocess Y_t with a single point catalyst at c.

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2. Let

$$u(t,z) := -\log \mathbb{E}_{\delta_z}\Big[\exp\Big(-\int_0^t \langle \phi, Y_s\rangle\, ds - \langle \varrho, Y_t\rangle\Big)\Big].$$

Unique Solution to the Control Problem

1. Consider the catalytic superprocess Y_t with a single point catalyst at c.
2. Let

$$u(t,z) := -\log \mathbb{E}_{\delta_z}\Big[\exp\Big(-\int_0^t \langle \phi, Y_s\rangle\, ds - \langle \varrho, Y_t\rangle\Big)\Big].$$

3. Recall that $\mathscr{X}$ denotes the class of all progressively measurable control processes ξ for which $\int_0^T |\xi_t|\, dL_t < \infty$ P_z-a.s. for all $T > 0$ and $z \in \mathbb{R}$.

Theorem (N. and Schied, 2016)

Let $\beta := 1/(p-1)$ and

$$\xi_t^* := -x_0 \exp\Big(-\int_0^t u(T-r, S_r)^\beta \, dL_r\Big) u(T-t, S_t)^\beta$$

so that

$$X_t^{\xi^*} = x_0 \exp\Big(-\int_0^t u(T-r, S_r)^\beta \, dL_r\Big).$$

Then ξ^* is the **unique strategy** in $\mathscr{X}$ minimizing the cost functional. Moreover, the minimal cost is given by

$$C([0,T]) = |x_0|^p u(T, z).$$

Connection between Control and Superprocesses

1. Super-Brownian motion $\{Y_t\}_{t\geq 0}$ satisfies

$$E_\mu[e^{-\langle Y_t,\phi\rangle}] = e^{-\langle v(r,\cdot),\mu\rangle},$$

for every test function ϕ.

Connection between Control and Superprocesses

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$$E_\mu[e^{-\langle Y_t,\phi\rangle}] = e^{-\langle v(r,\cdot),\mu\rangle},$$

   for every test function ϕ.

2. The log-Laplace functional v satisfies

$$\frac{\partial v}{\partial t} = \frac{1}{2}\Delta v - v^2, \quad v|_{t=0+} = \phi.$$

Connection Between Control and Superprocesses

In [Schied, 2013] the following value function was introduced

$$V(t, z, x_0) := \inf_{x()} E_{t,z}\Big[\int_t^T |\dot{x}(u)|^2 du + \int_t^T |x(u)|^2 a(W_u) du\Big].$$

1. Here W is a standard Brownian motion and a is some positive measurable function.

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1. Here W is a standard Brownian motion and a is some positive measurable function.
2. **The infimum is taken over the class of all absolutely continues adapted strategies $x()$ such that $x(t) = x_0$ and $x(T) = 0$.**

Connection Between Control and Superprocesses

$$V(t,z,x_0) := \inf_{x()} E_{t,z}\Big[\int_t^T |\dot{x}(u)|^2 du + \int_t^T |x(u)|^2 a(W_u) du\Big].$$

The associated HJB equation is

$$V_t(t,z,x_0) + \inf_{\zeta}\{|\zeta|^2 + V_{x_0}(t,z,x_0)\zeta\} + a(z)|x_0|^2 + \frac{1}{2}\Delta V(t,z,x_0) = 0.$$

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with $V(T,z,x_0) = 0$ if $x_0 = 0$ and $V(T,z,x_0) = \infty$ otherwise.

Connection Between Control and Superprocesses

The associated HJB equation is

$$V_t(t,z,x_0)+\inf_{\zeta}\{\zeta^2+V_{x_0}(t,z,x_0)\zeta\}+a(z)|x_0|^2+\frac{1}{2}\Delta V(t,z,x_0)=0,$$

with $V(T,z,x_0)=0$ if $x_0=0$ and $V(T,z,x_0)=\infty$ otherwise.

For $x_0>0$, assume that $V(t,z,x_0)=x_0^2 v(t,z)$ for some function v.

Connection between control problems and superprocesses

If we minimize over ζ we get that v formally satisfies:

$$\begin{cases} \frac{\partial v}{\partial t} = -\frac{1}{2}\Delta v + v^2 - a, \\ \\ v(T,z) = +\infty. \end{cases}$$

The Log-Laplace functional of SBM with branching rate 1 satisfies

$$\frac{\partial v}{\partial t} = \frac{1}{2}\Delta v - v^2, \quad v|_{t=0+} = \phi.$$


THANK YOU
for your
ATTENTION!